Differences in Tranching Methods: Some Results and Implications

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We review the mathematics of evaluating the credit risk of tranches of structured transactions with simple loss-priority structure for two common tranching approaches: PD-based tranching where the probability of default of a tranche is the quantity of interest; and EL-based tranching where the expected loss on a tranche is the quantity of interest. While comparing the attributes of these two tranching approaches, we examine their relative (level of credit enhancement) conservatism. While the mathematics is simple some implications of the results are interesting. We discuss the impact of the *collateral* LGD assumption on attachment points; show that, all else equal, lowering attachment point or the detachment point necessarily increases the tranche EL; and provide upper-bounds on senior-most *tranche* LGD under reasonable distributional assumptions. One implication of the low LGDs associated with the senior tranches is that under some EL definitions, it may be impossible to create a tranche with a given EL under the EL-based approach, even though they are always possible under the PD-based approach.

1 Introduction[1]

We review the mathematics of evaluating the credit risk of tranches of structured transactions with simple loss-priority structure for two common tranching approaches: PD-based tranching where the probability of default of a tranche is the quantity of interest; and EL-based tranching where the expected loss on a tranche is the quantity of interest.[2] These approaches are used to either evaluate the credit quality of an exogenously defined tranche or to determine the theoretically appropriate attachment and detachment points for a tranche to meet an exogenously defined target PD or EL. We compare the attributes of different tranching approaches.

[1] We are grateful to John Hull, Alan White, David Lando, Jordan Mann, Yufeng Ding, and Aziz Lookman for extensive comments on earlier drafts of this article. We also thank all of the members of Moody's Academic Research and Advisory Committee for useful discussions during presentations of this work at MAARC meetings. We wish to thank Shirish Chinchalkar for extensive programming work to generate the numerical examples and for his comments. All errors are of course, our own. Finally, the views presented here are those of the authors and do not represent the views of our current or former employers.

[2] Please note that we abuse the notation a bit to use probability of default (PD) to denote the probability of an economic loss on the tranche since default for a tranche as a legal or contractual term may not be very well-defined.

We also examine the relative conservatism of different approaches to tranching. We define tranching framework *X* as more "conservative" than tranching framework *Y* in a specific setting if, in that setting, tranche credit enhancement (CE) levels are higher when using framework *X* than when using framework *Y*. For our discussions, we assume that the "target" or idealized levels of EL and PD are exogenously defined.

We collect a series of sometimes disparate observations that follow simply from the mathematics of calculating PD and EL for a given tranche. Though these observations follow logically, a number of results could seem to be counter-intuitive. To illustrate, we use stylized collateral loss distributions, though the results hold for more realistic loss distributions as well. Our goal is to provide more transparency into the credit risk analytics of structured transactions as well as to provide some clear mathematical results and implications that market participants can use in thinking about these risks.

We show that (a) neither approach results in uniformly more or less conservative tranche levels;[3] (b) for a fixed detachment (attachment) point of a tranche, lowering the attachment (detachment) point necessarily increases the EL of the tranche, even if the tranche is made thicker by this move and regardless of the probability distribution of the underlying collateral; (c) because different organizations use different target PD and EL levels for determining risk grades and use different assumptions in generating these targets, it is typically unclear (without reference to the underlying loss distribution) which tranching method will imply higher CE levels in any given instance; (d) there is generally no constant LGD assumption (including 100% LGD) that can be made on *collateral* to translate a PD-based tranche into an EL-based tranche; (e) assuming a monotonically decreasing density of the loss distribution in the tail, the upper bound on the LGD of the senior-most tranche is 50% under EL-based tranching (in practice, for loss distributions often used by market participants, the LGD of the senior-most tranche may be much lower); (f) in the EL-based approach, it may be impossible to create a tranche with a target EL, given a specific capital structure and loss distribution; this will occur whenever the PD of the tranche above is higher than the target EL of the given tranche.

The remainder of this paper is organized as follows: Section 2 outlines notation and terminology. Section 3 briefly discusses the basic mathematics of tranching and some implications, including the observation that EL-based and PD-based tranching approaches do not produce similar attachment and detachment points for any tranche in the capital structure. Section 4 shows that, for a fixed detachment (attachment) point of a tranche, lowering the attachment (detachment) point necessarily increases the EL of the tranche regardless of the probability distribution of the collateral. Section 5 presents the upper-bound for the LGD on the senior-most tranche. Section 6 illustrates why some tranches with a target EL are unattainable under the EL-based tranching approach, even though they can exist under the PD-based approach.

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[3] This does not mean that one approach is "better" than the other. Rather, analyses that focus on EL vs. PD answer different questions, both of which are of interest to investors. Provided the analyses are used in the manner defined (i.e., as either an EL or a PD), there is no inconsistency.

2 Defining a tranche

A tranche is specified by *attachment* and *detachment* points of a tranche to collateral losses. These points partition losses on the underlying collateral such that losses below the attachment point do not affect the tranche, whereas losses above the attachment point are absorbed entirely by holders of the tranche until these losses exceed the detachment point, when the tranche has lost everything.

Tranching is the structured finance analog to the use by a corporation of multiple classes of liabilities. In both cases, the mechanism seeks to provide different liability holders with different liens on the assets supporting the liabilities should the issuer default. The function of tranching is to apportion losses in the underlying collateral loss distribution among tranche holders in a manner that provides more or less risk of loss to the holders of the different tranches. The most common tranching methods used target a specific PD (PD-based tranching) or a specific EL (EL-based tranching) in which the attachment and detachment points are set to achieve a target expected default frequency (PD) or expected loss (EL) for the tranche in order to appeal to investors with a specific risk preference. Clearly, in the settings above, tranche attachment and detachment points depend critically on the assumptions of the collateral loss distribution.[4]

Conversely, given a set of *exogenously* defined tranches and a loss distribution for the collateral, one can use the same analytic machinery to determine the PD or EL for a tranche. This is often done in the rating process where the capital structure of a transaction is defined by bankers or other structurers and ratings are then determined by raters, based on the credit enhancement implicit in the capital structure of the transaction. (e.g., Lucas, Goodman, and Fabozzi (2006)).

For purposes of our discussion here, we consider tranching in a very simple setting, similar to that which is done for synthetic transactions, by assuming no cash flow waterfall and a simple loss-priority approach. Assuming no waterfall simplifies the analysis but it ignores some features of cashflow timing that can affect tranching for cashflow transactions. Despite this limitation, many of our results carry over to the waterfall case. (Mahadevan, et. al (2006), for example, provides an overview of some typical securitization structures.)

The mathematics of tranching is relatively simple and involves only basic calculus and probability theory that can be found in any undergraduate textbook on probability. (e.g., Feller, W. (1978)). However, the subtlety lies primarily in the interpretation and the logical consequences of these results. In the next section we review the mathematical machinery in detail before going on to explore these consequences.

[4] The discussion of *which* distributional assumptions are appropriate for which assets and of *how* to parameterize these distributions is a widely debated topic that is beyond the scope of this article. For our purposes, we assume a collateral loss distribution that is given and may have been calculated analytically, through simulation, empirically or otherwise.

3 The mathematics of tranching

We examine two tranching approaches used to determine the attachment point, A, for a tranche with a given detachment point of D. Under both approaches, when the collateral pool underlying a transaction experiences a loss (L) that is less than A, the tranche experiences zero loss, whereas the tranche is wiped out if the pool losses exceed D. For simplicity, we assume that the pdf of the collateral loss distribution is given.

3.1 PD-based tranching

Tranche PD is the probability of collateral losses exceeding the attachment point. A tranche meets a given PD target if the tranche PD is less than an exogenously defined value, PD_T:

$$\begin{aligned} Tranche\,PD &= P(L > A) \\ &= \int_A^1 f(L)\cdot dL \\ &\le PD_T \end{aligned} \tag{1.1}$$

where

A ≡ the tranche attachment point

L ≡ the percentage loss on the portfolio

$f(\cdot)$ ≡ the pdf of the percentage loss on the portfolio, and

PD_T ≡ predefined target default rate for a tranche.

PD-based credit enhancement (CE or *subordination*) is equal to the collateral portfolio credit VaR[5] with $\alpha = PD_T$.

Note that the tranche width, *(D-A)*, does not appear anywhere in this expression, underscoring the insensitivity of the PD-based approach to the width of the tranche (or the severity of losses on it). Thus a very thick tranche and a very thin tranche, with the same attachment point, would have the same credit quality under the PD-based tranching approach. Thus, the required subordination to achieve a given target PD is the same for all variations on the size of the tranche and configuration capital structure of the transaction.[6]

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[5] e.g., Bohn and Stein (2009).

[6] Recently, there has been much attention focused on so called “thin tranches” that span only a small region of the loss distribution. These tranches are viewed by some as being more risky that traditional tranches since when they default, they can be completely wiped out very quickly. The PD approach treats thin tranches and thicker tranches the same with respect to subordination levels.

In the PD-based tranching approach, tranche credit enhancements can be calculated simply by determining the $1\text{-}PD_T$ th quantile of the loss distribution.[7]

3.2 EL-based tranching

For EL-based tranching the EL of the tranche is calculated. This calculation is more involved as it depends not only on the underlying loss distribution but also on the width of the tranche. Consider two tranches with identical attachment points but different detachment points. These tranches will experience different percentage losses for each dollar lost, since the loss will represent a different percentage of the total size of the respective tranches.

As in the case of PD-based tranching, whenever the pool experiences losses less than A, the tranche is unaffected. If losses are greater than D, the tranche is completely wiped out, i.e., it experiences its maximum loss, $D - A$. However, when the losses are between A and D, the measure is affected differentially: in absolute terms, the tranche loss increases linearly from *0* (at A) to $D - A$ (at D), or, in percentage terms, relative to the tranche par, from 0% at A to 100% at D. This can be expressed as

$$Percentage\ Tranche\,Loss = \min\left[1, \max\left(\left(\frac{L-A}{D-A}\right), 0\right)\right]$$

The expected loss of this tranche, relative to the tranche width $(D - A)$, can be expressed (cf., Pykhtin (2004)), using the pdf of the collateral loss distribution, as

$$\begin{aligned} Tranche\,EL &= \int_0^1 Percentage\ Tranche\,Loss \cdot f_L(L)\,dL \\ &= \int_0^1 \min\left[1, \max\left(\left(\frac{L-A}{D-A}\right), 0\right)\right] \cdot f_L(L)\,dL \end{aligned} \qquad (1.2)$$

where the terms are defined as above and

L $\equiv$ percentage loss on the portfolio

$f_L(\cdot)$ $\equiv$ pdf of the loss on the portfolio

Note that the tranche width, (D-A), is explicit in this expression, underscoring the sensitivity of the EL-based approach to the width of the tranche (and the severity of losses on it). It can be shown that, for the same attachment point, A, the smaller the width of the tranche (the closer we make D to A), the higher the tranche EL. (We provide more detail on this point in Section 5.)

[7] For some distributions, this can be done analytically. For empirical distributions, it can be done non-parametrically by simply choosing the quantile.

We can also use (1.2) to define a tranche, rather than to evaluate the credit quality of an exogenously defined tranche, in an iterative process. The process starts with the most senior tranche in the capital structure and repeats iteratively to the lowest rated tranche as follows:

First, the subordination level for the senior-most tranche is determined based on a desired target expected loss, EL_{Sr}. Because the senior-most tranche, by definition, is at the top of the capital structure, the detachment point, D, for that tranche is defined as100%. Therefore, the above formula for the senior-most tranche EL, with pool loss expressed as percent, can be used to define the Credit Enhancement for the senior-most tranche:

$$CE_{Sr} = \min\left\{ A \middle| \left[EL_{Sr} \geq \int_0^1 \min\left[1, \max\left(\left(\frac{L-A}{1-A}\right), 0\right)\right] \cdot f_L(L)\, dL \right]\right\}$$

We can easily use numerical methods to solve for lowest value of A that satisfies the above equation. This value of A is the credit enhancement, CE_{Sr}, which satisfies the credit quality criterion EL_{Sr}.

If there is an additional tranche subordinated to the senior-most tranche, we can now determine the subordination required for this tranche (Senior Subordinated) to achieve its desired target EL, EL_{SrSub}. The detachment point for the Senior Subordinated tranche is equal to the attachment point of the senior-most tranche. Thus, we obtain the Senior Subordinated attachment point by solving for lowest value of B that satisfies the expression below:

$$CE_{SrSub} = \min\left\{ B \middle| \left[EL_{SrSub} \geq \int_0^1 \min\left[1, \max\left(\left(\frac{L-B}{CE_{Sr}-B}\right), 0\right)\right] \cdot f_L(L)\, dL \right]\right\}$$

This process is repeated until all desired tranche levels are determined.

Unlike PD-based tranching, an attachment and detachment point cannot be found for every arbitrary EL target. In some cases, the shape of the distribution is such that some EL targets cannot be achieved through subordination without reducing the size of (or eliminating) tranches above. We explore this issue in greater details in Section 6.

Note also that from Equation (1.2) it is clear that assumptions about the LGD of the *collateral* (LGD_c), do not translate linearly into assumptions about the *tranche EL* LGD_t).

Also note that setting $LGD_c = 1$ (which makes the loss distribution equivalent to a default distribution) does *not* convert an EL-based tranching approach to a PD-based tranching approach. Two tranches with identical attachment points but different detachment points will have the same tranche PD but different *Tranche EL*s even if they are both based on the identical loss distribution and even if $LGD_c = 1$.

4 The EL of a tranche necessarily increases when either the attachment point or the detachment point is decreased

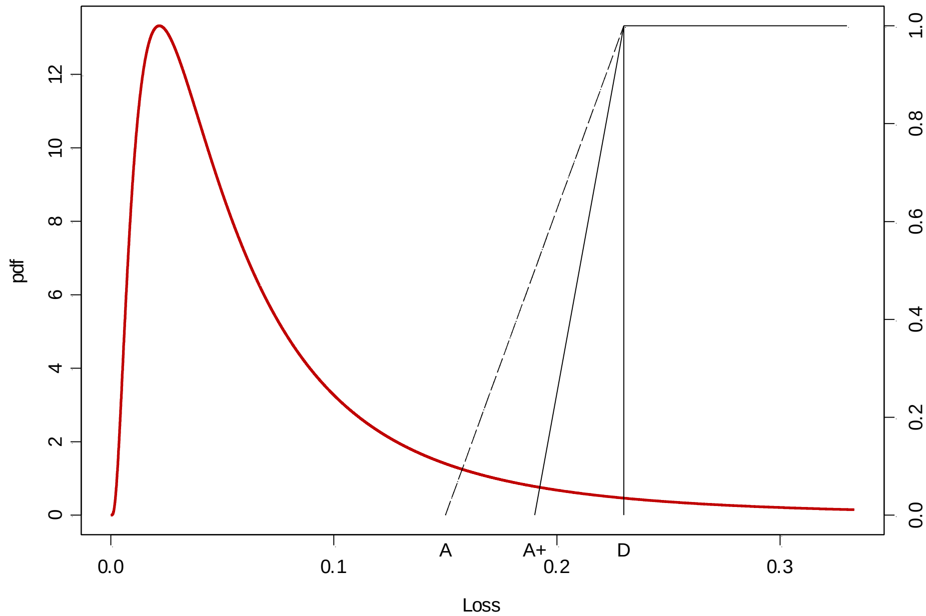
In this section, we show that, regardless of the probability density function of the underlying collateral, lowering the attachment point of a tranche while keeping the detachment point fixed necessarily increases the tranche EL.[8] We later show that decreasing the detachment point similarly necessarily increases the tranche EL.

Some observers erroneously reason that even though lowering the attachment point increases the PD, the tranche LGD could decrease since the new tranche is now thicker due to the new lower attachment point. In fact, there is some empirical evidence supporting the negative correlation between tranche size and tranche LGD. (Tung, Hu, and Cantor (2006)) Thus, they reason, it is not obvious whether EL would increase or decrease. This reasoning turns out to be flawed.

We now show that *regardless of the probability distribution of the collateral losses*, decreasing the attachment point or the detachment point necessarily increases the tranche EL. Although this argument can be made entirely geometrically (see Figure 1 and Figure 2) we will prove it analytically after making the geometric argument. First, we show that for a fixed detachment point, D, lowering the attachment point, A, increases EL.

Figure 1

[8] Throughout this note, we will make a rather innocuous technical assumption that the pdf of the collateral loss is continuous and positive everywhere. This allows us to make statements like “EL will necessarily increase” as opposed to “weakly increase,” which will be true for any collateral loss pdf.



In Figure 1, the horizontal axis shows the percentage collateral loss for a hypothetical loss distribution. Note that the horizontal axis should go from 0% to (not shown above) 100%. The left vertical axis shows the probability density function of the collateral losses while the right vertical axis shows the percentage loss on the tranche from 0 to 100%. In this figure, *D* denotes the detachment point of a tranche.

Consider two different cases of tranching for the same collateral pool: (1) when the attachment point, A^+, is higher; The plus superscript is used to remind us that this attachment point is above attachment point *A* and (2) when the attachment point is *A*.

The dashed line starting at *A* is the graph of percentage loss on the tranche as a function of collateral loss if *A* is the attachment point, and the solid line starting from A^+ is the graph of percentage loss on the tranche as a function of collateral loss if A^+ is the attachment point. We will show that, given a fixed detachment point, *D*, the EL for the tranche with attachment point, *A*, (case 2) is always as high as or higher than the EL for the tranche with attachment point, A^+, (case 1) regardless of the pdf of the collateral loss.

We now compute the EL for each case. For case 1 where the attachment point is A^+, the higher of the two attachment points, the tranche EL using equation (1.2) is:

$$\begin{aligned} Tranche\,EL_{A^+,D} &= \int_0^A 0\cdot f_L(L)\,dL + \int_A^{A^+} 0\cdot f_L(L)\,dL + \int_{A^+}^{D}\left(\frac{L-A^+}{D-A^+}\right)\cdot f_L(L)\,dL + \int_D^1 1\cdot f_L(L)\,dL \\ &= \int_{A^+}^{D}\left(\frac{L-A^+}{D-A^+}\right)\cdot f_L(L)\,dL + [1-F(D)] \end{aligned} \tag{1.3}$$

where $L \equiv$ the percentage loss on the collateral,

$F(D) \equiv$ the cumulative distribution function of the losses on the collateral, and other terms are as previously defined.

Similarly for Case 2, where the attachment point is *A*, the lower of the two attachment points, the tranche EL is given as:

$$\begin{aligned} Tranche\, EL_{A,D} &= \int_0^A 0 \cdot f_L(L)\,dL + \int_A^{A^+} \left(\frac{L-A}{D-A}\right) \cdot f_L(L)\,dL + \int_{A^+}^{D} \left(\frac{L-A}{D-A}\right) \cdot f_L(L)\,dL + \int_D^1 1 \cdot f_L(L)\,dL \\ &= \int_A^{A^+} \left(\frac{L-A}{D-A}\right) \cdot f_L(L)\,dL + \int_{A^+}^{D} \left(\frac{L-A}{D-A}\right) \cdot f_L(L)\,dL + \left[1 - F(D)\right] \end{aligned} \tag{1.4}$$

Comparing equations (1.3) and (1.4) we see that the last term is the same and the first term in (1.4) is non-negative. Hence, in order to show that the EL in (1.4) is higher than EL in(1.3), it is sufficient to show that

$$\int_{A^+}^{D} \frac{L-A}{D-A} \cdot f_L(L)\,dL \geq \int_{A^+}^{D} \frac{L-A^+}{D-A^+} \cdot f_L(L)\,dL$$

Since the pdf is non-negative, it is sufficient to show that

$$\left(\frac{L-A}{D-A}\right) \geq \left(\frac{L-A^+}{D-A^+}\right) \text{ for } L \in \left[A^+, D\right]$$

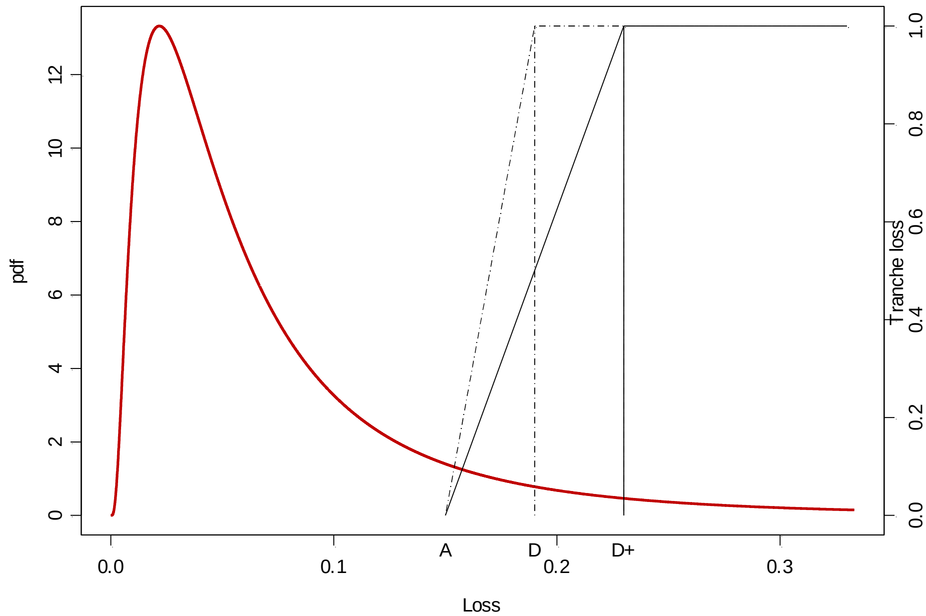
These two fractions are simply the percentage loss on the tranche in each case. Thus, we can prove the inequality by showing that the *percentage* tranche loss is as large or larger when the attachment point is *A* as compared to A^+. It is obvious from Figure 1 that the percentage tranche loss case 1 (shown in dotted lines) is as high or higher than that for case 2 everywhere. While the figure shows this geometrically, we can prove the inequality analytically by rewriting the first term of the above inequality as below and arguing that this term decreases as *A* increases.

$$\left(\frac{L-A}{D-A}\right) = 1 - \left(\frac{D-L}{D-A}\right)$$

Thus, we have shown that, for a fixed detachment point lowering the attachment point necessarily increases the EL of the tranche.

Using similar arguments to those just given, it is straightforward to see that as the detachment point, *D*, of a tranche is decreased but the attachment point, *A*, is kept constant, EL of the tranche increases as well. This is because the percentage losses on the tranche with lower detachment point, *D*, is as high as or higher than that for the tranche with the higher detachment point, *D+*, regardless of the level of the collateral loss. This can be easily seen in Figure 2.

Figure 2



Thus, using similar arguments to those used in the case of the attachment point, it can be shown that regardless of the probability distribution of the collateral, lowering the detachment point necessarily increases the EL of the tranche.

An implication of this result is that simply “thickening” a tranche does not necessarily make it less risky from an expected loss perspective.

5 Upper bound on tranche expected LGD (LGD_t) assumption given EL-based tranches

Consider now the *expected loss given default*[9] of a tranche with attachment point *A* and detachment point *D*. For losses between *A* and *D*, the percentage loss to the tranche is the probability weighted value of the tranche loss (*L-A*) as a percentage of the tranche size (*D-A*). Since we only consider losses above *A*, LGD_t is calculated by normalizing by the probability distribution in the region above *A* (the PD of the tranche) only. Finally, for collateral losses above *D*, the tranche loss is 100%, so the LGD contribution from this region is just 1 times the probability of losses above *D*, again normalized by the probability of losses above *A*. Mathematically, this can be written as:

[9] Tranche LGD here is not the same thing as LGD in the context of a corporate bond. With a corporate (or sovereign or municipal) bond, a default is a legally or contractually defined event, and an estimate of the monetary loss is made given that such an event occurs. Here, the tranche LGD is the estimate of the loss not conditional on a legal event but conditional on there being any loss at all to the tranche.

$$LGD_t = \frac{\int_A^D \left(\frac{L-A}{D-A}\right) \cdot f_L(L)\,dL + \int_D^1 1 \cdot f_L(L)\,dL}{\int_A^D f_L(L)\,dL + \int_D^1 f_L(L)\,dL} \quad (1.5)$$

First consider the LGD_t of a tranche which is not the senior-most. It is obvious that in the limit when the tranche is infinitesimally thin (*D* is very close to *A*), LGD_t becomes 1 since the first terms in both the numerator and the denominator collapse to zero. Similarly, as the tranche becomes wider the expected LGD of the tranche decreases from 1.

Now consider LGD_t for the senior-most tranche. As the detachment point is 1, equation (1.5) can be simplified to:

$$LGD_{Aaa} = \frac{\int_A^1 \left(\frac{L-A}{1-A}\right) \cdot f_L(L)\,dL}{\int_A^1 f_L(L)\,dL} \quad (1.6)$$

The tranche expected LGD for the senior-most tranche will depend on the probability density function of the collateral loss distribution in the region to the right of the attachment point for the senior-most tranche. Under most realistic assumptions the probability density function of the collateral loss value is non-increasing with extreme losses (i.e., for very large losses, the probability of a specific loss decreases as the losses get larger and larger)[10]. Under these plausible conditions, we show that the upper bound for expected LGD is 0.5.

Consider the case in which the pdf of the collateral loss value is constant beyond the attachment point. This represents the upper bound on the mass in the tail, given our assumption of a non-increasing tail. In this case, $f_L(L)$ equals $f_L(A)$ for $L \geq A$. Now (1.6) becomes:

$$LGD_{Sr} = \frac{\int_A^1 \left(\frac{L-A}{1-A}\right) \cdot f_L(A)\,dL}{\int_A^1 f_L(A)\,dL} = \frac{\int_A^1 \left(\frac{L-A}{1-A}\right) \cdot dL}{\int_A^1 dL} = \frac{1}{(1-A)^2} \cdot \int_A^1 (L-A) \cdot dL = 0.5$$

Thus, for non-increasing tails, the maximum possible mass in the tail results in $LGD_t=0.5$. Note again that we have assumed the maximum possible non-increasing tail

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[10] Admittedly this assumption may be questioned. For instance, one can imagine a distribution in which losses decline monotonically in the tail up to a point, but then spike for some reason. In such a setting, the results in this section would not hold.

here. In many collateral loss distributions the LGD_t value would be much smaller than 0.5 since the tail decreases and is convex.[11]

We illustrate the magnitude of tranche LGDs in the tail using a hypothetical loss distribution modeled as a *t*-copula with degrees of freedom equal to 4 (Cherubini (2004)). Note that we do not necessarily advocate the use of this assumption, and only use it to illustrate our results.

Table 1 below gives some examples of Senior tranche expected LGDs under this loss distribution assumption. (This first step is similar in some ways to Gregoriou and C. Hoppe (2008)). The table shows sample loss distributions using PD estimates for the collateral asset of 50, 100, and 200bps and correlation assumptions of 5, 15 and 25%. For this example, we assume the target tranche EL, EL_{Sr}, is equal to 5bps.

Table 1

PD (bp)	Rho	Sr Tranche attachment point (%)	Sr Tranche PD (bp)	Sr Tranche Expected LGD (%)
50	0.05	14.74	61	8.21
100	0.05	20..90	59	8.46
200	0.05	27.26	58	8.65
50	0.15	19.25	45	11.19
100	0.15	27.35	42	11.99
200	0.15	35.98	39	12.95
50	0.25	24.55	33	14.97
100	0.25	35.27	29	17.36
200	0.25	45.85	28	18.18

From the table, it is clear that, under a variety of parameterizations of the collateral loss distribution, tranche expected LGDs are well below the 50% upper bound. For corporate issuers, there is empirical evidence of a positive relationship between PD and LGD (e.g., Meng *et.al.* (2006)). However, in Table 1, we have imposed the condition that the EL of the Senior Tranche is constant at 5 bp. Since EL is the product of PD and LGD, the typical PD-LGD relationship is reversed and the tranche PDs vary inversely with expected LGD levels.

Conceptually, the lower the consequences of default (in terms of investor losses), the more default risk is tolerable given a constant target EL. This is reflected in attachment points that increase with the tranche expected LGD as well. We can contrast this with the PD-based approach which imposes the same PD assumption regardless of whether the risk in default to the investor is very high or very low.

An EL-based tranche will be less conservative (lower credit enhancement) than the PD-based tranche if the EL-based target is derived by taking the target PD (from a

[11] Note that the higher the convexity of the pdf in the region to the right of the attachment point, *A*, the lower the LGD of the tranche.

PD-based approach) and multiplying it by an LGD that is in the range ([tranche LGD], 1]. This relationship is formally shown below.

$$\text{Tranche } PD_{PD-based} \cdot LGD_{PD\_to\_EL\_multiplier} \equiv \text{Tranche } EL_{EL-based} = \text{Tranche } PD_{EL-based} \cdot \text{Tranche expected } LGD_{EL-based}$$

$$\text{Now, if } LGD_{PD\_to\_EL\_multiplier} > \text{Tranche expected } LGD_{EL-based}$$

$$\Rightarrow \text{Tranche } PD_{EL-based} = \frac{\text{Tranche } PD_{PD-based} \cdot LGD_{PD\_to\_EL\_multiplier}}{\text{Tranche expected } LGD_{EL-based}} > \text{Tranche } PD_{PD-based}$$

$$\Rightarrow \text{Tranche } CE_{EL-based} < \text{Tranche } CE_{PD-based}$$

In practice EL targets are not set by simply taking the PD targets and multiplying them with a constant LGD. In fact, industry practice is to use different target levels based on rating grades. These levels are provided by rating agencies and other market providers.

Thus if, for the same credit grade, organization A defines a target PD of 15 bps for its PD-based tranching approach, while organization B uses a target EL of 1.5bp, the relative conservatism of the approaches is unclear without reference to the actual loss distribution. In this case, for distributions in which the LGD of the tranche (either PD-based or EL-based) turns out to be less than 10%, the CE implied under the PD-based approach will be higher than the CE implied under the EL approach. However, for all other cases (i.e., where the LGD of the tranche is greater than 10%), the EL-based tranching approach will produce higher CE-levels.

6 "Skipping" of some tranches in the EL-based approach

Given a specific capital structure and loss distribution, under the EL-based approach, it may be impossible to create a tranche with a target EL; This will occur whenever the PD of the tranche above is higher than the target EL of the given tranche. To see this, consider equation (1.3) (reproduced here for convenience)

$$\begin{aligned} Tranche\, EL_{A^+,D} &= \int_0^A 0 \cdot f_L(L)\,dL + \int_A^{A^+} 0 \cdot f_L(L)\,dL + \int_{A^+}^{D} \left(\frac{L-A^+}{D-A^+}\right) \cdot f_L(L)\,dL + \int_D^1 1 \cdot f_L(L)\,dL \\ &= \int_{A^+}^{D} \left(\frac{L-A^+}{D-A^+}\right) \cdot f_L(L)\,dL + \left[1 - F(D)\right] \end{aligned}$$

Here, since the first term is non-negative, it is impossible to find an attachment point, A, such that Tranche EL is less than the target EL *whenever* the term [1 - $F(D)$] is higher than the target EL. The term [1 - $F(D)$] can be interpreted as the PD of the tranche (say, Senior) immediately above the one of interest (say, Senior Subordinated). This follows naturally since, in such a case, the loss of the Senior Subordinated tranche is 100% if the pool losses are higher than the detachment point of this tranche.

In words, this can be rewritten as:

$$Tranche\, EL_{A,D} = [\text{contribution between } A \text{ and } D] + [PD \text{ of tranche above} \times 1]$$

Conceptually, the EL for the tranche is the portion due to defaults and partial losses given default between *A* and *D* plus the total loss in the part beyond *D*. Thus, if the PD of the Senior tranche is higher than the target EL of the Senior Subordinated tranche, it is impossible to find an attachment point, *A*, that will make the EL of the tranche lower than the idealized EL since the portion of the EL defined by mass *above* the detachment point of the target tranche is *already* higher than the full target EL for the tranche.

For example, imagine that, in addition to creating a senior-most tranche targeting investors seeking an instrument with an EL (credit quality) of 5bps, the banker originating this transaction also wished to create a senior subordinated tranche, below the senior tranche, to appeal to investors seeking a target EL of EL_{SrSub}. In this example, assume that EL_{SrSub} = 10bps. Assume further that the collateral loss distribution was the one shown in the first row of Table 1. Looking at the first row pool in Table 1, we see that the PD of the senior-most tranche is 61 bps. In this case, it would not be possible for this pool to support a structure with an EL of 10bp since the PD of the senior-most tranche is already higher than the 10bps and the loss for the Senior Subordinated tranche beyond the senior-most tranche attachment point is always 100%. Thus, a tranche targeting investors seeking this target EL would not be feasible without reducing the size of the senior-most tranche.

7 Conclusion

In this brief note, we focus on the mathematics of tranching and its implications under both the PD- and EL-based tranching approaches. We show that (a) because different organizations use different target PD and EL levels for determining risk grades, it is typically unclear (without reference to the underlying loss distribution) which tranching method will imply higher CE levels; (b) for a fixed detachment (attachment) point of a tranche, lowering the attachment (detachment) point will necessarily increase the EL of the tranche regardless of the probability distribution of the underlying collateral; (c) there is generally no constant LGD assumption that can be made on collateral to translate a PD-based tranche into an EL-based tranche; (d) in the EL-based approach, it can often happen that it is impossible to create a tranche with a target EL, given a specific capital structure and loss distribution; this will occur whenever the PD of the tranche above is higher than the idealized EL of the given tranche and (e) the upper bound on the LGD of the senior-most tranche under many distributions of collateral is 50%. In many models commonly used in practice the LGD of the senior-most tranche will be lower than 50%.

Many of these observations lead naturally from our examination of tranche LGD, which itself follows from the width of the tranche. Simply put, the PD-based approach is not sensitive to the LGD of the tranche and is thus not sensitive to the size of the tranche. All equal, the rating under the PD approach will be the same, regardless of how thin the tranche is.

It is instructive to consider the differences in tranching approaches and which measures should appeal for which types of applications. First, we note that neither measure is “too conservative” or “too liberal.” By definition, each tranching approach delivers attachment points that are consistent with its objectives and definition. Further, because organizations that use these measures define their targets differently, it is typically unclear which will imply a higher credit enhancement level.

Finally, these measures should appeal to different users for different applications. For users concerned with never experiencing a default (even a default with *de minimus* economic impact) the PD-based measure is a more natural one. For users focused on managing economic loss, the EL measure may be more appropriate. For users concerned with conservatism, the choice is less clear since either measure may produce a more conservative estimate depending on the capital structure, collateral loss distribution, tranche of interest and definition of the target measure.

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